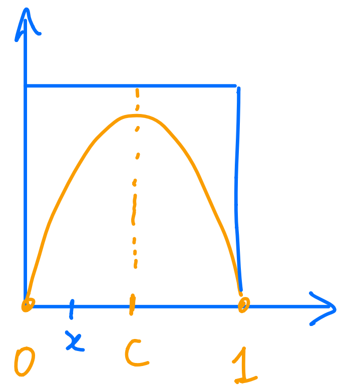
Lecture 18 – Holomorphic Dynamics

Kneading Theory (Milnor–Thurston)

Let $f: [0,1] \longrightarrow [0,1]$ a unimodal map with critical point c



Question: How do you compute the entropy of f?

Kneading Series & Determinant

Def.: The kneading coordinate of $x \in I = [0,1]$, $f^i(x) \neq c$ $\forall i \geq 0$, is $\vartheta(x) = (\vartheta_i(x))_{i \geq 0} \in \{+1, -1\}^{\mathbb{N}}$ defined as:

$$\vartheta_0(x) := \begin{cases} +1 & \text{if } x < c \\ -1 & \text{if } x > c \end{cases}$$

$$\vartheta_i(x) := \vartheta_0(f^i(x))\, \vartheta_{i-1}(x)$$

Note: $\vartheta_i(x) = \mathrm{sign}\left(\left(f^{i+1}\right)'(x)\right)$ by the chain rule.

$$\vartheta(x^+) := \lim_{y\to x^+} \vartheta(y)$$

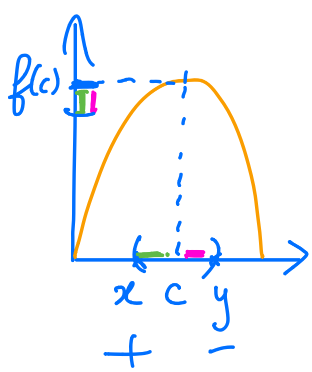
$$\vartheta(x^-) := \lim_{y\to x^-} \vartheta(x)$$

Def.:

$$\vartheta(x,t) := \sum_{i=0}^{\infty} \vartheta_i(x)\, t^i$$

Note.: for any x, $\vartheta(x,t)$ converges for $\{|t|<1\}$.

Note.: $\vartheta(c^-) = \lim_{x\to c} \vartheta(x)$ $\qquad$ $\vartheta(c^+) = \lim_{y\to c} \vartheta(y)$



$$\lim_{x\to c^-} \vartheta_0(f^i(x)) = \lim_{y\to c^+} \vartheta_0(f^i(y))$$

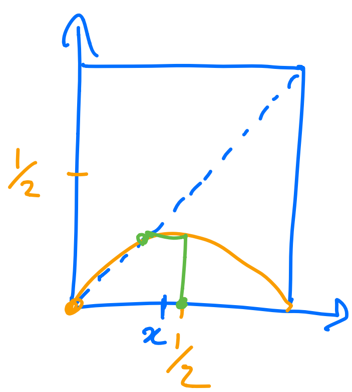
if $i \geq 1$

while $\vartheta_0(c^-) = -\vartheta_0(c^+)$

Hence $\vartheta(c^-, t) = -\vartheta(c^+, t)$.

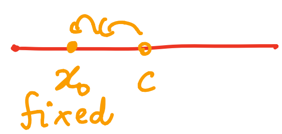
Def.: The kneading determinant of f is $D(t) := \vartheta(c^-, t)$.

Examples



$f_a(x) = ax(1-x)$

$f_a(\frac{1}{2}) = \frac{a}{4} < \frac{1}{2} \Rightarrow a <$



$\varepsilon_k(x) := \vartheta_0(f^k(x))$

$\varepsilon(c^-) = (+1, +1, +1, +1, +1, +1, +1)$

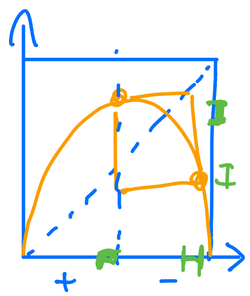
$\varepsilon(c^+) = (-1, +1, +1, +1, +1, +1, +1)$

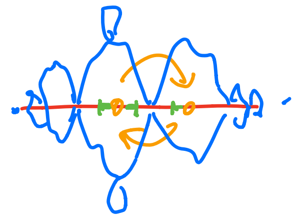
$\vartheta(c^-) = (+1, +1, +1, +1, \ldots)$

$\vartheta(c^+) = (-1, -1, -1, -1, \ldots)$

$D(t) = 1 + t + t^2 + \ldots = \dfrac{1}{1-t}$

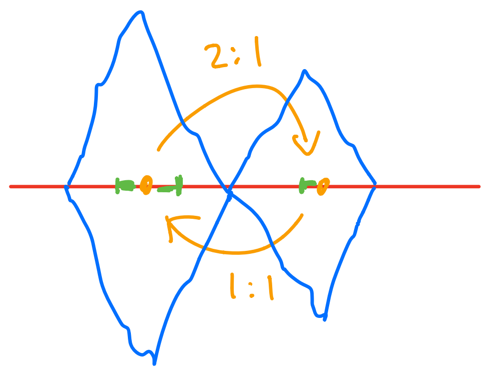
Basilica



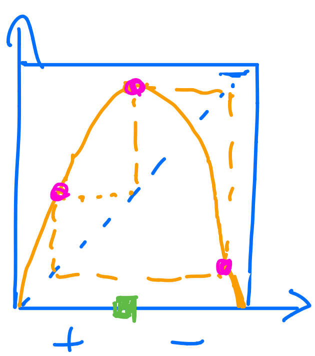


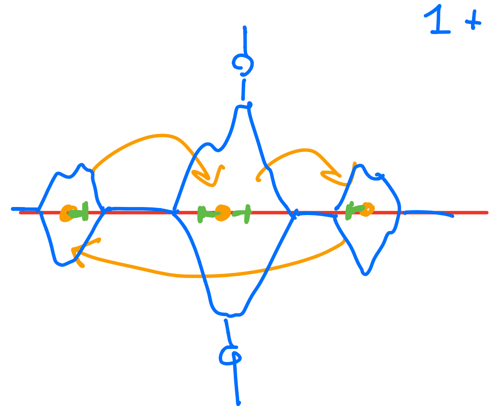
$$\varepsilon(c^-) = (+, -, -, -, -, \dots)$$

$$\vartheta(c^-) = (+, -, +, -, +, \dots)$$



$$D(t) = 1 - t + t^2 - t^3 + \dots = \frac{1}{1+t}$$





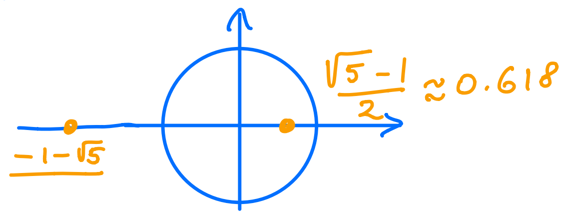
$$\varepsilon(c^-) = (+, -, +, -, -, +, -, -, +, \dots)$$

$$\vartheta(c^-) = (+, -, -, +, -, -, +, -, -, \dots)$$

$$D(t) = 1 - t - t^2 + t^3 - t^4 - t^5 + t^6 \cdots$$

$$= \frac{1 - t - t^2}{1 - t^3}$$

$\{ D(t) = 0 \}$



$$1 - t - t^2 = 0$$
$$t^2 + t - 1 = 0 \qquad t = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Theorem

Let f be a unimodal map, let $h = h_{top}(f)$ and let $D(t)$ be its kneading determinant. Then:

1. if $h = 0$, $D(t) \neq 0$ for $\{ |t| < 1 \}$.
2. if $h > 0$, then $h_{top}(f) = \log\left(\frac{1}{r}\right)$ where r is the smallest zero of $D(t)$ (which in fact is real).

Example f = airplane

$r = \frac{\sqrt{5}-1}{2}$, $h_{top} = \log\left(\frac{1}{r}\right) = \log \frac{\sqrt{5}+1}{2}$.